Chimera states: An unexpected phenomenon in the synchronization of networks

Introduction into the topic

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1.1 Theoretical background

Coupled oscillators: Chimera states are phenomena from the field of complex oscillatory systems. The entities of such systems are called oscillators because of their repeating behavior. Each oscillator has an inner state which evolves periodically. The inner state returns to the value that started after the period T. Since this is the case, the trajectory in the state space can be seen as a circle, with all possible states being dots on the circle. Embedding the state space as a unit circle with radius 1 into a complex plane allows us to treat the state of an oscillator as a complex number $z = e^{\imath\phi(t)}$ ($\imath$ is the complex unit). The phase $\phi(t)$ is the angle between the x-axis and the vector pointing to the complex number z. It evolves on the interval $[0, 2\pi]$ as shown in figure 1.1 with frequency ω.

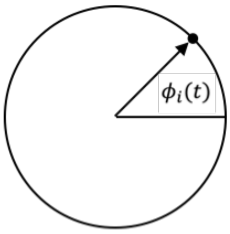

Figure 1.1: Circular representation of the phase $\phi_i(t) \in [0, 2\pi]$

The local behavior of an isolated oscillator follows simple rules and is easily predictable. However, if we consider a system of several coupled oscillators, the local oscillator and the global system behavior become more complex. In a group of $N > 1$ coupled oscillators, each oscillator reacts to the other oscillators' influences based on a coupling function. This function is a rule embedded in and followed equally by all oscillators. It determines how each oscillator has to adjust its phase when coupled with another oscillator besides the periodical development of its own phase. The dynamics that are enabled through the coupling of the oscillators lead to the emergence of collective behavior. The spontaneous synchronization of all oscillators' phases with initially incoherent states is one of the possible outcomes. Eventually, a system where collective behavior emerges from local rules is called self-organized. Famous examples of these phenomena are known from nature. Fireflies flashing in unison, cardiac pacemaker cells synchronously emitting electrical signals, and spontaneously synchronizing metronomes, among many other examples, have drawn the attention of different researchers and disciplines for decades [4, 8, 2, 6, 7].

Coupling forms: Complex oscillatory systems appear in different forms. A primary characteristic is the way of coupling, where we can distinguish between two cases: continuous and discrete. Continuous coupling refers to a system where each oscillator influences the others' phases at any point in time. Mechanical metronomes are continuously coupled, for example. These systems are described by means of differential equations.

On the other hand, discrete coupling represents discrete interactions where the oscillators send pulses to the others only at certain points in time. Discrete coupling is also called pulse-coupling. The oscillators' phases evolve periodically for both forms of coupling. However, the interaction between pulse-coupled oscillators (PCOs) only appears upon reaching the maximum value of the phase ϕ. Then the phase is reset to zero, and a pulsatile signal is sent to the other oscillators. This event is often called firing. Upon receiving a pulse, an oscillator adjusts its phase according to a phase response function $h(\phi)$. Fireflies, pacemaker cells, and neurons [1] are typical examples of PCOs.

Furthermore, the coupling can either be excitatory, inhibitory or both combined. Excitatory coupling refers to a signal which pushes the other oscillators' phases toward the maximum value of ϕ, whereas inhibitory coupling decreases the phases towards 0. A combination is given when the influence depends on the receiving oscillator's phase value at the moment of receipt.

Finally, the coupling can depend on the distance between the oscillators. We have global coupling if each oscillator is coupled to each other equally (Figure 1.2). Locally coupled oscillators only interact with nearest neighbors. Suppose the coupling strength decreases with the spatial distance between the oscillators. In that case, it is called non-local coupling (Figure 1.3), where close oscillators are strongly coupled, and wide apart oscillators have almost no influence on each other's phases.

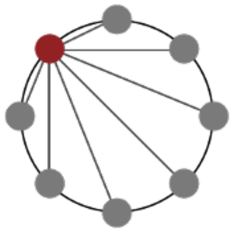

Figure 1.2: Global coupling

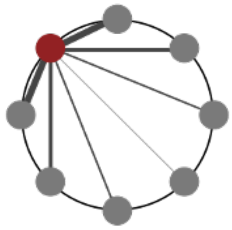

Figure 1.3: Non-local coupling

Furthermore, the coupling can be based on the links between oscillators in arbitrary forms of networks. In graph and network theory, the entities of a network are called nodes and the connections links. Only linked nodes can communicate with each other. Networks representing real systems are often modeled through random graphs. A random graph is obtained by successively adding links to the initially isolated nodes following a certain scheme. In the end, the connected nodes can either be coupled with equal strength, or the coupling strength decreases with the spatial distance, similarly to non-local coupling but only with regard to existing links. Typical examples of complex oscillatory systems modeled by Erdős-Rényi random graphs are neuronal networks [5].

Possible global states: A coupled oscillatory system can exhibit different global behaviors or states. These emerge based on the local interactions between the oscillators. A system is synchronous if the phase states of all oscillators are aligned. Phase-locked means that the phase gaps between the i^{th} and j^{th} oscillator are constant. A synchronous network can be phase-locked, i.e. – when all phase states are aligned but locked to a certain value and not evolving periodically. Chaos is given if the phases have no order and are not predictable. Finally, we have a chimera state if there is a simultaneous occurrence of chaotic and synchronous subsets of nodes within a single system.

1.2 Kuramoto's model

The original model of phase-coupled oscillatory systems: Kuramoto was the first to formulate a solvable differential equation for a continuously coupled oscillatory system [2] based on the previous work and simplifications of Winfree [8]. In Kuramoto's model, the oscillators are globally coupled, i.e., each oscillator influences each other equally according to the phase evolution equation

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \omega_i - \sum_{i=1}^{N} F_{ij}(\phi_i - \phi_j), \quad i = 1, ..., N \tag{1.1}$$

and the coupling function

$$F_{ij}(\phi_i - \phi_j) = \frac{k}{N} \sin(\phi_i - \phi_j). \tag{1.2}$$

The phase ϕ_i evolves depending on the natural frequency ω_i and is drawn from a random distribution. The coupling function F_{ij} in this case, is the sine of the phase difference between the i^{th} and j^{th} oscillator and the parameter k determines the strength of the coupling. This setting drives the frequencies towards a common frequency Ω if the coupling constant k is chosen high enough and if the natural frequencies ω_i are not too different from each other.

Depending on the specific conditions, the system can evolve into different states. Chaos is given when the phase gaps $\phi_i - \phi_j$ are not constant in time. This happens for low values of k and highly different frequencies ω_i. Furthermore, the system is phase-locked if the phase gap $\phi_i - \phi_j$ = const but not zero. If a system is phase-locked with a phase gap $\phi_i - \phi_j = 0$, we call it synchronized. Finally, partial synchrony appears if the system falls into synchrony, but a part of the oscillators remains unsynchronized because their natural frequencies differ too strongly from the mean. Chimera states do not occur under these circumstances.

The first chimera model: Kuramoto and Battogtokh were working on an adjusted model of non-locally coupled phase oscillators, which were identical in their frequencies, when they unexpectedly discovered a state in which a coherent and an incoherent subpopulation coexisted [3]. Before, it was believed that a population of oscillators with the same frequencies $\omega_i = \Omega$ = const can either remain incoherent or synchronize their phases. The coexistence of chaos and synchrony was believed to depend on the natural frequencies' heterogeneities.

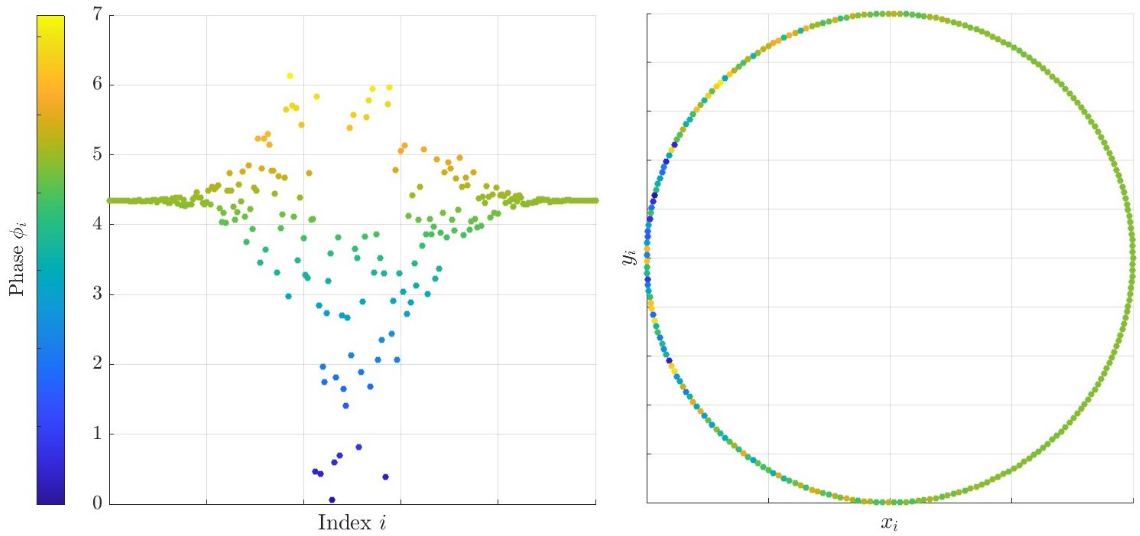

Figure 1.4: Chimera state: phases of oscillators on a circle (original model).

The simulation of the original model by Kuramoto with N = 256 phase-coupled oscillators on a circle results in a chimera state, with a synchronous and a chaotic subset of oscillators.

Based on the original model of Kuramoto, simulations are performed using the software MATLAB. The final phase distribution can be seen in figure 1.4. It shows the deterministic spatial distribution on a circle on the right side and the system's phase distribution on the left. On the right side, it can be seen that a one-headed chimera state emerged with a chaotic subset of oscillators that is smaller than the synchronous one.

The chimera state occurred unexpectedly due to a few changes to the original Kuramoto model, as can be obtained from the equation

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}t} = \Omega - \sum_{i=1}^{N} K_{ij} \sin(\phi_i - \phi_j + \alpha), \quad i = 1, ..., N. \tag{1.3}$$

First, as mentioned above, the natural frequencies are no longer drawn from a random distribution but are chosen to be equal. The phase gap α is added as a configuration parameter, for which certain values lead to a chimera state. Additionally, the initial phases are drawn from Gaussian distribution.

Second, the coupling function was changed. Kuramoto's first model has global coupling (Figure 1.2). The coupling function which finally leads to a chimera state is non-local (Figure 1.3). The coupling strength decreases exponentially with the distance between the oscillators, which is realized in the coupling kernel

$$K_{ij} = Ce^{-k \cdot d_{ij}} \tag{1.4}$$

and the distance function

$$d_{ij} = \min\left\{\frac{|i-j|}{N}, 1 - \frac{|i-j|}{N}\right\} \tag{1.5}$$

The constant C is used for normalization, and the coupling constant k determines the coupling strength. Third, another main difference is the oscillators' arrangement on a circle, unlike in Kuramoto's first model, where the spatial arrangement had no relevance due to the global coupling. The chimera phase pattern of identical non-locally coupled phase oscillators has two main differences compared to partial synchrony appearing in the original Kuramoto model: The oscillators' phases synchronize, not only their frequencies. Furthermore, due to the circular arrangement, it can be seen that the coherent and the incoherent oscillators are located in separate areas on the circle, split into two subsets of oscillators.

1.3 A new chimera model

The model presented in this sub-chapter has been developed in cooperation with Arke Vogell.

The new chimera model is a further developed pulse-coupled version of the continuously-coupled model by Kuramoto and Battogtokh [3]. First, the deterministic spatial distribution on a circle remains the same. The oscillators' positions x_i are chosen to be equivalent to the indices i, which are evenly distributed on the interval $[1, N]$ with N oscillators, thus representing a deterministic circular spatial distribution. The distance function

$$d_{ij} = \min\{|x_i - x_j|, N - |x_i - x_j|\}/N \tag{1.6}$$

returns values in the interval $[0, 0.5]$. The maximum distance of $d_{ij} = 0.5$ applies to a pair of oscillators located directly opposite each other on a circle. Second, the coupling remains non-local and is realized in the coupling kernel 1.4 where the normalization constant is set to $C = 1$ and the coupling constant to $k = 4$. Third, the oscillators' phases evolve linearly with $\frac{\mathrm{d}\phi}{\mathrm{d}t} = 1$ on the interval [0,1] (normalized by 2π). Upon reaching the value 1, a pulse is emitted and the phase is reset to 0. The phase response function $h(\phi_i)_j$ returns the i^{th} oscillator's phase upon receiving a pulse from the j^{th} oscillator and is defined as

$$h(\phi_i)_j = (\phi_i - K_{ij} \sin(2\pi\phi_i + \alpha)) \bmod 1. \tag{1.7}$$

Whereas the dynamics in the original, continuously coupled chimera model from [3] depend on the sine of the phase difference between two oscillators, the new phase response only depends on the spatial distance between them and the receiving oscillators' current phase ϕ_i. The sending oscillator's phase is always $\phi_j = 0$ at the moment of pulse emission. Furthermore, the phase gap is fixed at $\alpha = 1.47$, which causes a phase shift of the sine by almost $\pi/2$. Introducing the modulo 1 operation ensures that phases remain in the interval [0,1].

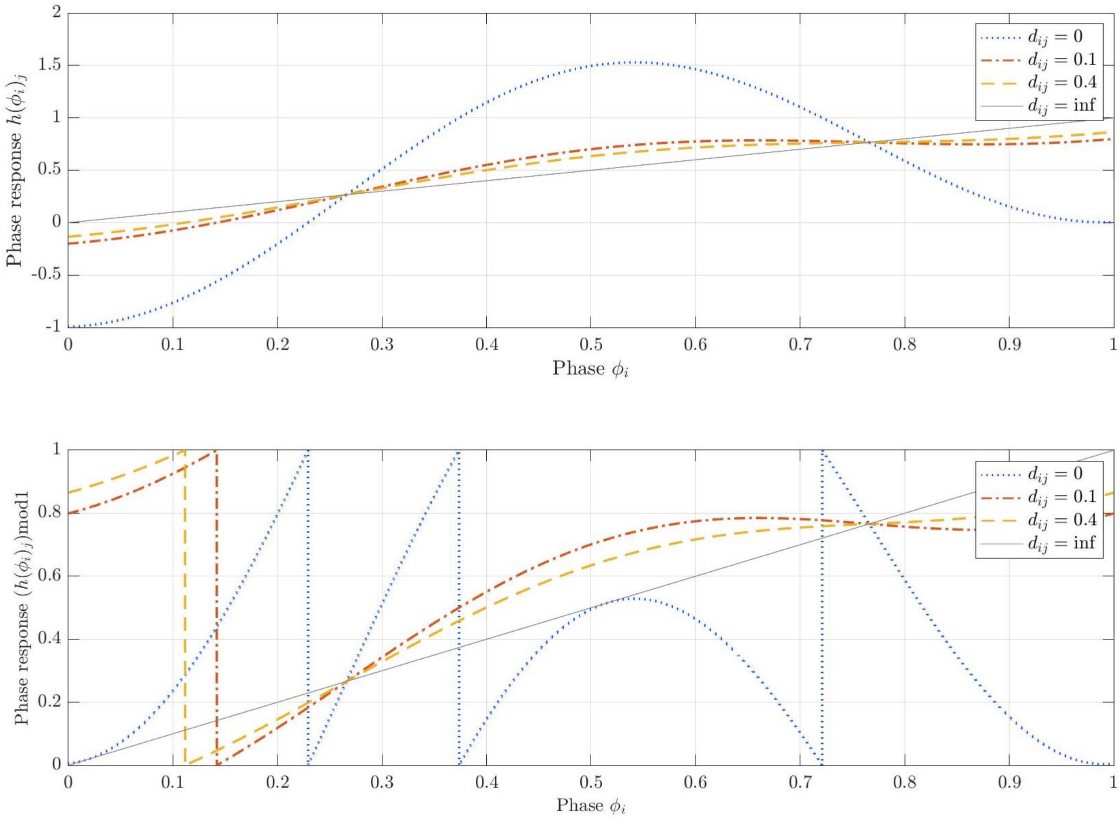

Figure 1.5: Phase response functions of the new model.

Phase response functions for pairs of oscillators with different distances for $k = 4$. The nodal points are fixed points of the dynamical system. The synchronizing phases are drawn towards the stable fixed point at $\phi_i = h(\phi_i)_j = 0.78$.

The two plots in figure 1.5 each show four curves of the phase response $h(\phi_i)_j$ depending on the distance d_{ij} between the emitting and the receiving oscillator. The upper (lower) plot is the function before (after) introducing the modulo operation. In both cases, the curves become flatter with increasing distance, and subsequently, the coupling strength decreases.

The plots reveal two fixed points of the dynamical system, which are independent of the modulo operation. The fixed points are located where all curves intercept with the identity. The identity is equal to $d_{ij} \to \infty$. If the distance between two oscillators is infinite, there is no coupling and $\phi_i = h(\phi_i)_j$ for any value of ϕ_i. The fixed point at $\phi_i = h(\phi_i)_j = 0.78$ is stable. This leads to the phenomenon that the synchronizing sub-group of oscillators is always drawn towards and phase-locked at this point.

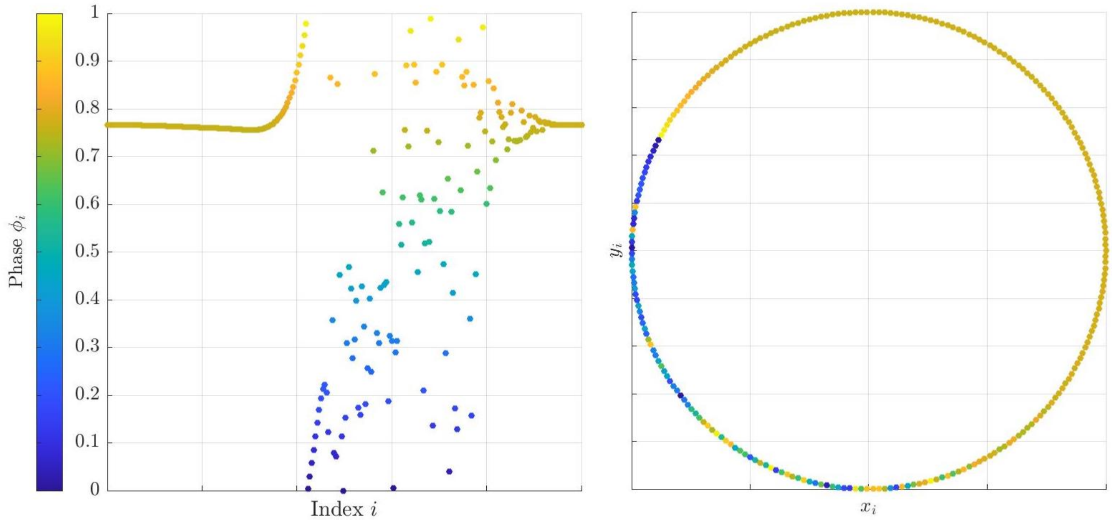

Figure 1.6: Chimera state: oscillators on a circle.

The simulation with $T = 10000$ and $N = 256$ oscillators results in a chimera state. The synchronous subset's phases are drawn toward the stable fixed point at $\phi_i = 0.78$ (left plot). The chaotic subset drifts along the circle during the simulation (right plot).

Application fields: Applications for chimera states may include interference and collision avoidance, wireless and sensor networks, production and logistics. Further analogies to similar phenomena in economics, business and nature can be drawn.

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